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On the Relationship between Killing Tensors and Killing–Yano Tensors

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Abstract

The conditions that a Killing tensor of order 2 be the contracted product of a Killing–Yano tensor of order 2 with itself are found. All Killing-Yano tensors of order 2 admitted in empty space-times are obtained

1. *Introduction*

A symmetric tensor a_{ij} satisfying the equation

$$a_{ij;k} + a_{jk;i} + a_{ki;j} = 0 \tag{1.1}$$

is said to be a Killing tensor of order 2 whilst a skew symmetric tensor f_{ij} satisfying the equation

$$f_{ij;k} + f_{ik;j} = 0 \tag{1.2}$$

is said to be a Killing-Yano tensor of order 2. A comprehensive review of the literature on such tensors up to 1973 is given by Dietz (1974), and more recent results are to be found in Collinson (1974) and Hauser and Malhiot (1975, 1976). The contracted product $f_{ik}f^k{}_j$ of a Killing-Yano tensor with itself is a Killing tensor, and Dietz states that the Killing tensor associated with the Kerr metric can be written as such a product. This has led the author to ask the following question: If a_{ij} is a given Killing tensor, what conditions must it satisfy in order that it can be written as the contracted product of a Killing-Yano tensor with itself? This question, which is analogous to the Rainich problem for the Einstein-Maxwell field (Rainich, 1925), is answered in Section 2. All Killing-Yano tensors admitted in empty space-times are found in Section 3 and the associated Killing tensors given. The null tetrad notation of Newman and Penrose (1962) is used throughout.

2. *The Conditions under Which a Killing Tensor Is the Contracted Product of a Killing-Yano Tensor*

Suppose a_{ij} is a given Killing tensor. We seek the conditions under which a_{ij} can be written as

$$a_{ij} = f_{ik}f^k{}_j \tag{2.1}$$

where the skew symmetric tensor f_{ij} satisfies equation (1.2). The first condition will be on algebraic condition on a_{ij} ensuring the existence of a tensor f_{ik} satisfying (2.1). This algebraic condition can best be given in terms of an invariant classification of second order symmetric tensors, for example the classification due to Collinson and Shaw (1972).

If f_{ij} is a null bivector then one can choose a null tetrad such that

$$f_{ij} = 2cl_{[i}m_{j]} + 2\bar{c}l_{[i}\bar{m}_{j]} \tag{2.2}$$

It then follows that

$$a_{ij} = 2c\bar{c}l_il_j \tag{2.3}$$

In terms of the Shaw classification $a_{ij} \in N1 + [4]$ with $\mu = 0$. Given a tensor a_{ij} of this form one can certainly write down a (nonunique) tensor f_{ij} such that (2.1) is satisfied. If (2.3) is substituted into the equation (1.1) then it is easily established that $(c\bar{c})^{1/2}l_i$ is a Killing vector so that a_{ij} becomes the direct product of a Killing vector with itself. Such degenerate Killing tensors are of no physical interest and the case when f_{ij} is a *null* bivector will be discussed no further.

If f_{ij} is a non-null bivector then one can choose a null tetrad such that

$$f_{ij} = 2al_{[i}n_{j]} + 2ibm_{[i}\bar{m}_{j]} \tag{2.4}$$

It then follows that

$$a_{ij} = 2a^2l_{(i}n_{j)} + 2b^2m_{(i}\bar{m}_{j)} \tag{2.5}$$

In terms of the Shaw classifications $a_{ij} \in D1\,[2, 2]$ with $\mu_1 = -\mu_4 \leqslant 0$. Given a tensor a_{ij} of this form one can write down a tensor f_{ij} satisfying equation (2.1). The tensor f_{ij} will be defined uniquely up to signs. This contrasts strongly with the corresponding result for an Einstein-Maxwell field where the energy momentum tensor is the *trace-free* part of the contracted product of the electromagnetic field tensor with itself. Then the field is only defined up to a duality rotation. The results proved so far can be stated as a theorem.

> *Theorem 1.* A given Killing tensor a_{ij} can be written in the algebraic form (2.1) if and only if $a_{ij} \in N1 + [4]$ with $\mu = 0$ or $a_{ij} \in Dl[2, 2]$ with $\mu_1 = -\mu_4 \leqslant 0$. In the first case the Killing tensor is necessarily degenerate.

Suppose now that the Killing tensor $a_{ij} \in D1\,[2, 2]$ with $\mu_1 = -\mu_4 \leqslant 0$ so that a_{ij} can be written in the form (2.5). Then the null tetrad components of (1.1) can be written out explicitly to give

$$
\begin{aligned}
Da^2 &= \Delta a^2 = \delta b^2 = 0 \\
\delta a^2 &= (a^2 + b^2)(\bar{\pi} - \tau) \\
Db^2 &= -(a^2 + b^2)(\rho + \bar{\rho}) \\
\Delta b^2 &= (a^2 + b^2)(\mu + \bar{\mu})
\end{aligned}
\tag{2.6}
$$

and, provided that the Killing tensor is not proportional to the metric tensor g_{ij},

$$\kappa = \sigma = \lambda = \nu = 0 \tag{2.7}$$

The conditions (2.7) imply that the null vectors l_i and n_i are both tangent to shear-free geodesic congruences. With conditions (2.6) and (2.7) it can easily be shown that the tensor f_{ij} will be a Killing-Yano tensor, that is, will satisfy equation (1.2) provided that a and b can be chosen so that

$$
\begin{aligned}
a(\tau + \bar{\pi}) &= ib(\tau - \bar{\pi}) \\
a(\rho + \bar{\rho}) &= ib(\rho - \bar{\rho}) \\
a(\mu + \bar{\mu}) &= ib(\mu - \bar{\mu})
\end{aligned}
\tag{2.8}
$$

Hence we have the following theorem:

> *Theorem 2.* A Killing tensor $a_{ij} \in D1\,[2, 2]$ with $\mu_1 = -\mu_4 \leqslant 0$ can be written as the contracted product of a Killing-Yano tensor with itself if and only if the signs of a and b can be chosen so that the equations (2.8) are satisfied.

Notice that the equations (2.8) are consistent (with not both a and b zero) if and only if

$$\tau\bar{\rho} = \bar{\pi}\rho, \qquad \tau\bar{\mu} = \bar{\pi}\mu, \qquad \rho\bar{\mu} = \bar{\rho}\mu$$

With these the only other condition that need be imposed on the Killing tensor a_{ij} for it to be the contracted product of a Killing-Yano tensor with itself is that a^2 and b^2 must satisfy the equation obtained by squaring any one nontrivial equation of the set (2.8).

3. *Killing-Yano Tensors of Order 2 in Empty Space-Times*

The author has already (Collinson, 1974) discussed the integrability conditions for the equation (1.2) and has proved that in empty space-times Killing-Yano tensors can only exist if the space-times are of Petrov types N or D. The Killing-Yano tensor for the type N empty space-times was shown to be a null bivector and so the associated Killing tensor is degenerate. The Killing-Yano tensors for the type D Robinson-Trautman space-times and for the N.U.T. space-times were found. This work can easily be extended to all type D empty space-times by using the classification of such space-times given by Kinnersley (1969). It is then found that the only type D empty space-time that does *not* admit a Killing–Yano tensor is the space-time IIIB. For the space-times I, II, and IIIA the Killing–Yano tensor is defined uniquely (up to a constant multiplicative factor) and can be written as

$$f_{ij} = \frac{i(\rho - \bar{\rho})}{\rho\bar{\rho}} (l_i n_j - l_j n_i) + \frac{i(\rho + \bar{\rho})}{\rho\bar{\rho}} (m_i \bar{m}_j - m_j \bar{m}_i)$$

For the space times IV (corresponding to $\rho = 0$) the Killing–Yano tensor can be written as

$$f_{ij} = x(l_i n_j - l_j n_i) - ia(m_i \bar{m}_j - m_j \bar{m}_i)$$

Here the notation of Kinnersley is used. The associated Killing tensors are

$$a_{ij} = \frac{(\rho - \bar{\rho})^2}{(\rho\bar{\rho})} (l_i n_j + l_j n_i) + \frac{(\rho + \bar{\rho})^2}{(\rho\bar{\rho})} (m_i \bar{m}_j + m_j \bar{m}_i)$$

and

$$a_{ij} = x^2(l_i n_j + l_j n_i) + a^2(m_i \bar{m}_j + m_j \bar{m}_i)$$

These Killing tensors, except in the case of the space-time IIIA, have been found previously by Hughston and Sommers (1973).

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